Pointwise energy conditions

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Chapter 1

Pointwise energy conditions

In this section we review the main pointwise energy conditions used within the context of general relativity.

Energy conditions are additional, coordinate-independent restrictions placed on the tensors appearing in general relativity. Energy conditions are additional, coordinate-independent restrictions placed on the tensors appearing in general relativity. One may begin with an arbitrary metric, calculate the corresponding Einstein tensor $G_{\mu\nu}$, and then define the stress–energy tensor by

$$T_{\mu\nu} := \frac{1}{8\pi} G_{\mu\nu}, \tag{1.1}$$

thereby obtaining a formal solution to Einstein's field equations. However, the resulting stress–energy tensor may have highly exotic or physically unreasonable properties. The purpose of the energy conditions is therefore to restrict the wide range of stress–energy tensors and spacetime geometries that are mathematically compatible with Einstein's equations by expressing assumptions about what constitutes physically reasonable matter and energy.

Pointwise energy conditions impose such restrictions separately at every spacetime point. They require a specified contraction of a rank-two tensor with causal vectors to be non-negative. The different conditions are distinguished by the tensor combination involved and by whether the vectors are timelike or null.

Each condition may be presented in three equivalent formulations. The **physical formulation** is written in terms of the stress–energy tensor

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.2}$$

then provide a bridge to the **geometric formulation**, in which the same restriction is expressed using the Einstein tensor $G_{\mu\nu}$ or the Ricci tensor $R_{\mu\nu}$. Finally, when the stress–energy tensor is of Hawking–Ellis Type I form[[1], p. 89],

$$T^{\hat{\mu}\hat{\nu}} = \mathrm{diag}\,(\rho, p_1, p_2, p_3) \tag{1.3}$$

(where $T^{\hat{\mu}\hat{\nu}}$ are the components of the stress–energy tensor with respect to that orthonormal basis) the condition can be written in an **effective formulation** as inequalities involving the energy density ρ and the principal pressures p_i. This form makes the physical meaning of the tensor inequalities easier to interpret.

The four main classical pointwise energy conditions are the **Weak**, **Strong**, **Dominant** and **Null Energy Conditions**. The following sections state each condition in its physical, geometric and effective formulations and discuss its interpretation.

The Weak Energy Condition (WEC)

To an observer whose world-line at **p** has unit tangent vector t^μ, the local energy density ppears to be $T_{\mu\nu}t^\mu t^\nu$. The Weak Energy Condition may be stated in its physical, geometric and effective forms as follows:

Weak Energy Condition (WEC)

physical	$T_{\mu\nu}t^\mu t^\nu \geq 0$	for all timelike future-pointing vector t^μ
geometric	$G_{\mu\nu}t^\mu t^\nu \geq 0$	for all timelike future-pointing vector t^μ
effective	$\rho \geq 0$ and $\rho + p_i \geq 0$	$i = (1, 2, 3)$

In this case, the effective formulation provides a straightforward interpretation of the physical form. It tells us that an observer travelling on a timelike curve always measures the energy density from all the matter fields in the universe to be non-negative. Also, there is no restriction on the sign of the pressure but, if negative, its magnitude cannot be bigger than that of the energy density. It is, however, much more challenging to find an interpretation of the geometric form. This is because there is no intuitive understanding of the Einstein tensor in geometric terms. Unlike the Riemann tensor, which is easily associated with geodesic deviation, there is no similar interpretation that captures the geometric significance of $G_{\mu\nu}$.

By continuity the condition $T_{\mu\nu}t^\mu t^\nu \geq 0$ will then also be true for any null vector.

The Strong Energy Condition (SEC)

The Strong Energy Condition may be stated in its physical, geometric and effective forms as follows:

Strong Energy Condition (SEC)

physical	$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)t^\mu t^\nu \geq 0$	for all timelike future-pointing vector t^μ
geometric	$R_{\mu\nu}t^\mu t^\nu \geq 0$	for all timelike future-pointing vector t^μ
effective	$\rho + p_i \geq 0$ and $\rho + \sum_{i=1}^{3} p_i \geq 0$	$i = (1, 2, 3)$

Here, the effective formulation tells us that in case of having negative pressures, the energy density has to be larger than the magnitude of the momentum flux in each of the three spacelike directions. It also tells us that the sum of the pressures in the three directions cannot negatively dominate the energy density at one specific spacetime point.

Choosing the observer's rest frame reduces the physical formulation of the SEC to

$$T_{00} - \frac{1}{2}T \geq 0. \tag{1.4}$$

This quantity is conventionally called the effective energy density, or historically the effective density of gravitational mass. This terminology labels the relevant contraction but does not, by itself, provide a particularly illuminating physical interpretation of it.

To obtain the geometric form from the physical one, we have taken EFEs and multiplied both sides by the metric, thus finding the expression $-R = 8\pi T$. Then, one can rewrite EFEs as

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{1.5}$$

and we use this to bridge the physical and geometric formulations of the SEC.

The geometric form of the Strong Energy Condition is

$$R_{\mu\nu}t^\mu t^\nu \geq 0 \tag{1.6}$$

for every timelike future-directed vector t^μ. Its interpretation may be obtained from the equation of geodesic deviation, which has already been derived and interpreted in the main text. Recall that, for an affinely parametrised timelike geodesic $\gamma(\tau)$ with tangent vector t^μ,

$$R^\rho{}_{\mu\nu\sigma}t^\mu \xi^\nu t^\sigma = t^\mu \nabla_\mu (t^\nu \nabla_\nu \xi^\rho) = \frac{D^2\xi^\rho}{d\tau^2}. \tag{1.7}$$

Here, ξ^μ is a connecting vector from γ to an infinitesimally nearby geodesic in the same congruence. It is taken to satisfy

$$\xi^\mu t_\mu = 0, \qquad \mathcal{L}_t \xi^\mu = 0, \tag{1.8}$$

at the point under consideration. Since t^μ is timelike and ξ^μ is orthogonal to it, ξ^μ is spacelike. As explained previously, the right-hand side of the geodesic-deviation equation represents the relative acceleration of the two neighbouring geodesics.

Taking the radial component and averaging over the three spatial directions gives the average radial acceleration (see Appendix A)

$$A_r := -\frac{1}{3}\xi_r t^\mu \nabla_\mu \left(t^\nu \nabla_\nu \xi^r\right). \tag{1.9}$$

Using the geodesic-deviation equation, this may be written as

$$A_r = -\frac{1}{3}R_{\mu\nu} t^\mu t^\nu. \tag{1.10}$$

Because A_r is a scalar, it gives a coordinate-independent measure of the average relative radial acceleration of neighbouring timelike geodesics. If the SEC holds, then

$$R_{\mu\nu} t^\mu t^\nu \geq 0, \tag{1.11}$$

and therefore

$$A_r \leq 0. \tag{1.12}$$

Thus, neighbouring timelike geodesics do not accelerate apart on average. The geometric interpretation of the SEC is consequently that congruences of timelike geodesics are focused: they converge, or at least do not diverge.

The Dominant Energy Condition (DEC)

The Dominant Energy Condition may be stated in its physical, geometric and effective forms as follows:

Dominant Energy Condition (DEC)		
physical	$T_{\mu\nu} t^\mu \xi^\nu \geq 0$	for all co-oriented timelike future-pointing vectors t^μ and ξ^μ
geometric	$G_{\mu\nu} t^\mu \xi^\nu \geq 0$	for all co-oriented timelike future-pointing vectors t^μ and ξ^μ
effective	$\rho \geq 0$ and $\lvert p_i \rvert \leq \rho$	$i = (1, 2, 3)$

Co-oriented means having the same time orientation: the vectors are either both future-directed or both past-directed. The DEC may also be formulated using a single future-directed timelike vector t^μ. For every such t^μ, one requires

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \tag{1.13}$$

and that the energy-momentum flux vector

$$-T^\mu{}_\nu t^\nu \tag{1.14}$$

be future-directed and non-spacelike. The first condition is the WEC, while the second ensures that the energy-momentum flux measured by any timelike observer does not propagate faster than light.

Let

$$J^\mu := -T^\mu{}_\nu t^\nu$$

be the energy–momentum flux associated with a future-directed timelike vector t^μ. Since $T_{\mu\nu}$ is symmetric,

$$T_{\mu\nu} t^\mu \xi^\nu = -J_\mu \xi^\mu.$$

For the signature $(-+++)$, a vector J^μ is future-directed and causal if and only if

$$J_\mu \xi^\mu \leq 0$$

for every future-directed timelike vector ξ^μ. Therefore,

$$T_{\mu\nu} t^\mu \xi^\nu \geq 0$$

for all future-directed timelike vectors t^μ and ξ^μ is equivalent to requiring that

$$-T^\mu{}_\nu t^\nu$$

be future-directed and causal for every future-directed timelike vector t^μ.

Taking $\xi^\mu = t^\mu$ gives

$$T_{\mu\nu} t^\mu t^\nu \geq 0,$$

which is the WEC. Conversely, the WEC fixes the future orientation of the flux vector, while the requirement that it be non-spacelike ensures that the energy–momentum flux is causal.

The physical and effective formulations imply that a timelike observer measures the energy density to be non-negative and the momentum flux to be causal and with the same orientation as the proper time of the observer. One can then think of the DEC as ruling out superluminal travel of any source of the stress-energy tensor. The geometric formulation suffers from the same difficulties as the WEC and therefore lacks a plain interpretation.

The Null Energy Condition (NEC)

The Null Energy Condition may be stated in its physical, geometric and effective forms as follows:

Null Energy Condition (NEC)		
physical	$T_{\mu\nu}k^\mu k^\nu \geq 0$	for all null vectors k^μ
geometric	$R_{\mu\nu}k^\mu k^\nu \geq 0$	for all null vectors k^μ
effective	$\rho + p_i \geq 0$	$i = (1, 2, 3)$

The first point to notice is that, for the NEC, Einstein's field equations relate the contraction of the stress–energy tensor directly to the corresponding contraction of the Ricci tensor. Since a null vector k^μ satisfies

$$g_{\mu\nu}k^\mu k^\nu = 0,$$

we have

$$8\pi T_{\mu\nu}k^\mu k^\nu = \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)k^\mu k^\nu = R_{\mu\nu}k^\mu k^\nu.$$

It follows that

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \qquad \Longleftrightarrow \qquad R_{\mu\nu}k^\mu k^\nu \geq 0.$$

For a Hawking–Ellis Type I stress–energy tensor, the effective formulation of the NEC is

$$\rho + p_i \geq 0, \qquad i = 1, 2, 3.$$

Unlike the WEC, the NEC does not have a direct interpretation as the energy density measured by a physical observer, since no observer can have a null four-velocity. Nevertheless, null directions are physically significant because they describe the propagation of light and other massless fields.

Its geometric interpretation is obtained in a manner analogous to that of the SEC, but now using a congruence of affinely parametrised null geodesics with tangent vector k^μ. As explained in the appendix, the relative acceleration of neighbouring geodesics is governed by the null equation of geodesic deviation. The physically relevant connecting vectors are taken in the two-dimensional screen space transverse to the null direction.

Averaging the transverse relative acceleration over the two independent screen-space directions gives

$$A_r = -\frac{1}{2}R_{\mu\nu}k^\mu k^\nu.$$

Therefore, if the NEC holds, then

$$R_{\mu\nu}k^\mu k^\nu \geq 0,$$

and hence

$$A_r \leq 0.$$

Thus, the NEC implies that null geodesic congruences focus, or at least do not defocus.

Appendix A

Equation of geodesic deviation

A.1 Derivation of the equation of geodesic deviation for time-like case

Let γ be a timelike geodesic belonging to a congruence of neighbouring timelike geodesics, and let t^μ be the tangent vector field to the geodesics of the congruence.

Write

$$x^\mu = x^\mu(\tau, \nu), \tag{A.1}$$

where τ is proper time along the geodesics and ν parametrises a curve connecting neighbouring geodesics. Define

$$t^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{A.2}$$

and

$$\eta^\mu = \frac{\partial x^\mu}{\partial \nu}. \tag{A.3}$$

Thus, t^μ is tangent to each timelike geodesic in the congruence, while η^μ is a connecting vector between neighbouring geodesics.

The commutator of t^μ and η^μ satisfies

$$\begin{aligned} [t,\eta]^\mu &:= t^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu t^\mu \\ &= \frac{\partial x^\nu}{\partial \tau}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^\mu}{\partial \nu}\right) - \frac{\partial x^\nu}{\partial \nu}\frac{\partial}{\partial x^\nu}\left(\frac{\partial x^\mu}{\partial \tau}\right) \\ &= \frac{\partial x^\mu}{\partial \tau \partial \nu} - \frac{\partial x^\mu}{\partial \nu \partial \tau} \\ &= 0. \end{aligned} \tag{A.4}$$

since partial derivatives commute.

Because the Levi-Civita connection is torsion-free, partial derivatives may be replaced by covariant derivatives in the Lie bracket:

$$\begin{aligned} 0 &= \mathcal{L}_t \eta^\mu \\ &= t^\nu \partial_\nu \eta^\mu - \eta^\nu \partial_\nu t^\mu \\ &= t^\nu \partial_\nu (\eta^\mu + \Gamma^\mu_{\nu\rho} \eta^\rho) - \eta^\nu (\partial_\nu t^\mu + \Gamma^\mu_{\nu\rho} t^\rho) \\ &= t^\nu \nabla_\nu \eta^\mu - \eta^\nu \nabla_\nu t^\mu. \end{aligned} \tag{A.5}$$

Here we have used

$$\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}.$$

Using the directional-derivative notation $\nabla_X := X^\nu \nabla_\nu$, equation (A.5) becomes

$$\nabla_t \eta^\mu = \nabla_\eta t^\mu. \tag{A.6}$$

Applying ∇_t to both sides gives

$$\nabla_t \nabla_t \eta^\mu = \nabla_t \nabla_\eta t^\mu. \tag{A.7}$$

Now consider the curvature identity

$$\nabla_X (\nabla_Y Z^\mu) - \nabla_Y (\nabla_X Z^\mu) - \nabla_{[X,Y]} Z^\mu = R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma. \tag{A.8}$$

Setting $X^\mu = Z^\mu = t^\mu$, and $Y^\mu = \eta^\mu$, the second term vanishes because t^μ is tangent to an affinely parametrised geodesic:

$$\nabla_t t^\mu = t^\nu \nabla_\nu t^\mu = 0. \tag{A.9}$$

The third term, $[t, \eta]^\nu \nabla_\nu t^\mu$, also vanishes by equation (A.4). Hence equation (A.8) becomes

$$\nabla_t \nabla_\eta t^\mu - R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \eta^\sigma = 0. \tag{A.10}$$

Substituting equation (A.7), we obtain

$$\nabla_t \nabla_t \eta^\mu - R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \eta^\sigma = 0. \tag{A.11}$$

By definition

$$\frac{D^2 \eta^\mu}{d\tau^2} := \nabla_t \nabla_t \eta^\mu,$$

the equation of geodesic deviation is therefore

$$\frac{D^2 \eta^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \eta^\sigma. \tag{A.12}$$

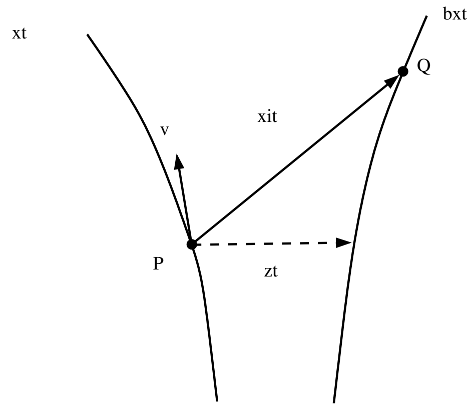

Figure A.1: ξ^μ is the orthogonal connecting vector.

We are interested only in the spatial part of the connecting vector and in the corresponding geodesic-deviation equation. We therefore define the orthogonal connecting vector ξ^μ by

$$\xi^\mu := h^\mu{}_\nu \eta^\nu, \tag{A.13}$$

where

$$h^\mu{}_\nu = \delta^\mu{}_\nu + t^\mu t_\nu \tag{A.14}$$

is the projection tensor onto the three-dimensional rest space orthogonal to t^μ. It follows that

$$t_\mu \xi^\mu = 0. \tag{A.15}$$

Since

$$\eta^\mu = \xi^\mu - t^\mu t_\nu \eta^\nu, \tag{A.16}$$

we have

$$\begin{aligned}
\frac{D\eta^\mu}{d\tau} &= t^\rho \nabla_\rho \eta^\mu \\
&= t^\rho \nabla_\rho \left(\xi^\mu - t^\mu t_\nu \eta^\nu\right) \\
&= t^\rho \nabla_\rho \xi^\mu - \left(t^\rho \nabla_\rho t^\mu\right) t_\nu \eta^\nu - t^\mu \left(t^\rho \nabla_\rho t_\nu\right) \eta^\nu - t^\mu t_\nu \left(t^\rho \nabla_\rho \eta^\nu\right) \\
&= t^\rho \nabla_\rho \xi^\mu \\
&= \frac{D\xi^\mu}{d\tau}.
\end{aligned} \tag{A.17}$$

Here we have used the geodesic equation $t^\rho \nabla_\rho t^\mu = 0$, together with $\nabla_t \eta^\mu = \nabla_\eta t^\mu$ and $t_\nu \nabla_\rho t^\nu = 0$.

Furthermore,

$$\begin{aligned}
R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \eta^\sigma &= R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \left(\xi^\sigma - t^\sigma t_\lambda \eta^\lambda\right) \\
&= R^\mu{}_{\nu\rho\sigma} t^\nu t^\rho \xi^\sigma,
\end{aligned} \tag{A.18}$$

because $R^{\mu}{}_{\nu\rho\sigma}$ is antisymmetric in its final two indices.

Consequently, the orthogonal connecting vector satisfies

$$\frac{D^2\xi^{\mu}}{d\tau^2} = R^{\mu}{}_{\nu\rho\sigma} t^{\nu} t^{\rho} \xi^{\sigma}, \tag{A.19}$$

This has the same form as equation (A.12), but ξ^{μ} now denotes the spatial connecting vector orthogonal to t^{μ}.

We now show that the orthogonal connecting vector ξ^{μ} is Lie transported along the timelike geodesics. Recall that

$$\xi^{\mu} = h^{\mu}{}_{\nu}\eta^{\nu} = \eta^{\mu} + t^{\mu} t_{\nu}\eta^{\nu},$$

where

$$[t, \eta]^{\mu} = \mathcal{L}_t \eta^{\mu} = 0.$$

Therefore,

$$\begin{aligned} \mathcal{L}_t \xi^{\mu} &= [t, \xi]^{\mu} \\ &= [t, \eta + (t_{\nu}\eta^{\nu})\, t]^{\mu} \\ &= [t, \eta]^{\mu} + [t, (t_{\nu}\eta^{\nu})\, t]^{\mu} \\ &= t^{\rho}\nabla_{\rho}(t_{\nu}\eta^{\nu})\, t^{\mu}. \end{aligned} \tag{A.20}$$

The scalar $t_{\nu}\eta^{\nu}$ is constant along each geodesic, since

$$\begin{aligned} t^{\rho}\nabla_{\rho}(t_{\nu}\eta^{\nu}) &= \left(t^{\rho}\nabla_{\rho} t_{\nu}\right)\eta^{\nu} + t_{\nu} t^{\rho}\nabla_{\rho}\eta^{\nu} \\ &= t_{\nu}\nabla_{\eta} t^{\nu} \\ &= \frac{1}{2}\eta^{\rho}\nabla_{\rho}(t_{\nu}t^{\nu}) \\ &= 0. \end{aligned} \tag{A.21}$$

Here we have used the geodesic equation

$$\nabla_t t^{\mu} = 0,$$

the relation

$$\nabla_t \eta^{\mu} = \nabla_{\eta} t^{\mu},$$

and the normalization

$$t_{\mu}t^{\mu} = -1.$$

It follows from equation (A.20) that

$$\mathcal{L}_t \xi^{\mu} = [t, \xi]^{\mu} = 0. \tag{A.22}$$

Thus, the orthogonal connecting vector ξ^{μ} is Lie transported along the timelike geodesic congruence.

A.2 Derivation of the equation of geodesic deviation for null case

Appendix B

Average Radial Acceleration

B.1 Average Radial Acceleration

We have already introduced the equation of geodesic deviation and the associated description of the relative acceleration of neighbouring geodesics in the main text. Here we use those ideas to define the average radial acceleration of a causal geodesic. Let u^μ be a future-directed causal vector field whose integral curves γ are affinely parametrised geodesics. If γ is timelike, we assume that u^μ is a unit vector field. Let ξ^μ be a vector field along γ such that, at one point,

$$\xi^\nu u_\nu = 0,$$

and

$$\mathcal{L}_u \xi^\mu = 0.$$

If u^μ is null, then ξ^μ may be proportional to u^μ; otherwise, ξ^μ must be spacelike. It follows that $\xi^\nu u_\nu = 0$ at every point of γ. As discussed in the Appendix A, ξ^μ may be regarded as a connecting vector field joining γ to a neighbouring, infinitesimally separated integral curve of u^μ, while

$$u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi^\mu \right)$$

represents the acceleration of that neighbouring geodesic relative to γ. By the equation of geodesic deviation,

$$u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi^\mu \right) = R^\mu{}_{\alpha\beta\gamma} u^\alpha \xi^\beta u^\gamma.$$

Now fix an orthonormal triad of connecting vector fields

$$\{\xi_i^\mu\}_{i \in \{1,2,3\}}$$

along γ. The magnitude of the radial component of the relative acceleration in the ith direction is

$$-\xi_{i\mu} \, u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi_i^\mu \right).$$

At a fixed point $p \in \gamma$, the average radial acceleration $\mathcal{A}_{\mathrm{r}}$ of γ is therefore defined by

$$\mathcal{A}_{\mathrm{r}} := -\frac{1}{k}\sum_i \xi_{i\mu}\, u^\alpha \nabla_\alpha\big(u^\beta \nabla_\beta \xi_i^\mu\big),$$

where $k = 3$ when u^μ is timelike and $k = 2$ when it is null. It is straightforward to verify that $\mathcal{A}_{\mathrm{r}}$ is independent of the choice of orthonormal triad. It therefore defines an intrinsic geometrical quantity associated with the tangent vector u^μ.

A straightforward calculation, presented in the next section using the equation of geodesic deviation, shows that

$$\mathcal{A}_{\mathrm{r}} = -\frac{1}{k} R_{\mu\nu} u^\mu u^\nu. \tag{B.1}$$

Thus, the contraction of the Ricci tensor with the tangent vector determines the average radial acceleration of neighbouring geodesics relative to γ.

If the Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu},$$

is assumed, equation (B.9) becomes

$$\mathcal{A}_{\mathrm{r}} = -\frac{8\pi}{k}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) u^\mu u^\nu. \tag{B.2}$$

For a null tangent vector, $g_{\mu\nu} u^\mu u^\nu = 0$ and $k = 2$, so this reduces to

$$\mathcal{A}_{\mathrm{r}} = -4\pi T_{\mu\nu} u^\mu u^\nu. \tag{B.3}$$

B.2 Details

B.2.1 Expressing inverse metric in terms of frame fields

Say we start with four lineraly independent vector fields ${e_I}^\mu$ where I labels the vectors.

Assume that g_{IJ} is singular. Then there exists a nonzero set of components V^J such that

$$g_{IJ} V^J = 0.$$

Using

$$g_{IJ} = g_{\mu\nu} {e_I}^\mu {e_J}^\nu,$$

we obtain

$$g_{\mu\nu} {e_I}^\mu {e_J}^\nu V^J = 0$$

for every I. Define the spacetime vector

$$W^\nu = e_J{}^\nu V^J.$$

Then

$$g_{\mu\nu} e_I{}^\mu W^\nu = 0$$

for every I. Now define the covector

$$\omega_\mu = g_{\mu\nu} W^\nu.$$

The preceding equation becomes

$$e_I{}^\mu \omega_\mu = 0$$

for every I. Since the vectors $e_I{}^\mu$ are linearly independent, we have

$$\omega_\mu = 0.$$

Therefore,

$$g_{\mu\nu} W^\nu = 0.$$

Since $g_{\mu\nu}$ is non-singular, it follows that

$$W^\nu = 0.$$

But

$$W^\nu = e_J{}^\nu V^J,$$

so

$$e_J{}^\nu V^J = 0.$$

Since the vectors $e_J{}^\nu$ are linearly independent, this implies

$$V^J = 0,$$

contradicting the assumption that V^J is nonzero. Therefore the kernel of g_{IJ} is trivial, and hence g_{IJ} is invertible.

Since g_{IJ} is invertible, its inverse g^{IJ} exists. We may therefore define

$$e^I{}_\mu := g^{IJ} g_{\mu\nu} e_J{}^\nu.$$

Then

$$\begin{aligned} e^I{}_\mu e_J{}^\mu &= g^{IK} g_{\mu\nu} e_K{}^\nu e_J{}^\mu \\ &= g^{IK} g_{JK} \\ &= \delta^I{}_J. \end{aligned}$$

Thus $e^I{}_\mu$ is a left inverse of the matrix $(e_I{}^\mu)$.

Moreover, the vectors $e_I{}^\mu$ are linearly independent, so the columns of the square matrix $(e_I{}^\mu)$ are linearly independent. Hence

$$\det(e_I{}^\mu) \neq 0,$$

and therefore $({e_I}^\mu)$ is invertible. A square matrix has a unique inverse, so the left inverse ${e^I}_\mu$ is also the right inverse. Consequently,

$$ {e_I}^\mu {e^I}_\nu = {\delta^\mu}_\nu . $$

Finally, using this together with

$$ {e^I}_\nu = g^{IJ} g_{\nu\rho} {e_J}^\rho , $$

we find

$$ \begin{aligned} {\delta^\mu}_\nu &= {e_I}^\mu {e^I}_\nu \\ &= {e_I}^\mu g^{IJ} g_{\nu\rho} {e_J}^\rho \\ &= \left(g^{IJ} {e_I}^\mu {e_J}^\rho \right) g_{\rho\nu} . \end{aligned} $$

Thus $g^{IJ} {e_I}^\mu {e_J}^\rho$ is the inverse of $g_{\rho\nu}$. Since the inverse metric is unique, it follows that

$$ g^{\mu\nu} = g^{IJ} {e_I}^\mu {e_J}^\nu . $$

If we take the tetrad to consist of one unit timelike vector u^μ and three orthonormal spacelike vectors ξ_i^μ the orthonormality relations lead to

$$ g_{IJ} = \eta_{IJ} = \mathrm{diag}\left(-1, 1, 1, 1\right) , $$

where the frame metric is the Minkowski metric η_{IJ}. We then have,

$$ g^{\mu\nu} = -u^\mu u^\nu + \sum_{i=1}^{3} \xi_i^\mu \xi_i^\nu . \tag{B.4} $$

We now take ${e_0}^\mu = u^\mu$ and ${e_1}^\mu = n^\mu$ where

$$ u^\mu u_\mu = n^\mu n_\mu = 0 $$

together with the normalisation condition

$$ u^\mu n_\mu = -1. $$

Then, if we take ${e_2}^\mu = \xi_2^\mu$ and ${e_3}^\mu = \xi_3^\mu$ to be orthonormal spacelike vectors satisfying ${e_i}^\mu u_\mu = {e_i}^\mu n_\mu = 0$ for $i = 2, 3$, the orthonormailty relations lead to the frame metric

$$
g_{IJ} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

We then have,

$$
g^{\mu\nu} = -u^\mu n^\nu - n^\mu u^\nu + \sum_{i=2}^{3} \xi_i^\mu \xi_i^\nu. \tag{B.5}
$$

B.2.2 Proving euqtion (B.1)

Substituting the equation of geodesic deviation into the definition of $\mathcal{A}_{\mathrm{r}}$ gives

$$
\begin{aligned}
\mathcal{A}_{\mathrm{r}} &= -\frac{1}{k} \sum_i \xi_{i\mu}\, u^\alpha \nabla_\alpha \left(u^\beta \nabla_\beta \xi_i^\mu \right) \\
&= -\frac{1}{k} \sum_i \xi_{i\mu} R^\mu{}_{\alpha\nu\beta} u^\alpha \xi_i^\nu u^\beta \\
&= -\frac{1}{k} R_{\mu\alpha\nu\beta} u^\alpha u^\beta \sum_i \xi_i^\mu \xi_i^\nu.
\end{aligned} \tag{B.6}
$$

First suppose that u^μ is timelike. Since u^μ is unit timelike and the three ξ_i^μ form an orthonormal basis of the subspace orthogonal to u^μ, the inverse metric decomposes as

$$
g^{\mu\nu} = -u^\mu u^\nu + \sum_{i=1}^{3} \xi_i^\mu \xi_i^\nu.
$$

Hence

$$
\sum_{i=1}^{3} \xi_i^\mu \xi_i^\nu = g^{\mu\nu} + u^\mu u^\nu.
$$

It follows that

$$
\begin{aligned}
R_{\mu\alpha\nu\beta} u^\alpha u^\beta \sum_{i=1}^{3} \xi_i^\mu \xi_i^\nu &= R_{\mu\alpha\nu\beta} u^\alpha u^\beta \left(g^{\mu\nu} + u^\mu u^\nu \right) \\
&= g^{\mu\nu} R_{\mu\alpha\nu\beta} u^\alpha u^\beta + R_{\mu\alpha\nu\beta} u^\mu u^\alpha u^\nu u^\beta.
\end{aligned} \tag{B.7}
$$

The second term vanishes because $R_{\mu\alpha\nu\beta}$ is antisymmetric in μ and α, whereas $u^\mu u^\alpha$ is symmetric. Using

$$
g^{\mu\nu} R_{\mu\alpha\nu\beta} = R_{\alpha\beta},
$$

we therefore obtain

$$R_{\mu\alpha\nu\beta} u^\alpha u^\beta \sum_{i=1}^{3} \xi_i^\mu \xi_i^\nu = R_{\alpha\beta} u^\alpha u^\beta.$$

Now suppose that u^μ is null. Choose a second null vector n^μ such that

$$n^\mu n_\mu = 0, \qquad u^\mu n_\mu = -1,$$

and choose two unit spacelike connecting vectors λ_1^μ and λ_2^μ orthogonal to both u^μ and n^μ. The inverse metric then decomposes as

$$g^{\mu\nu} = -u^\mu n^\nu - n^\mu u^\nu + \sum_{i=2}^{3} \xi_i^\mu \xi_i^\nu,$$

so that

$$\sum_{i=2}^{3} \xi_i^\mu \xi_i^\nu = g^{\mu\nu} + u^\mu n^\nu + n^\mu u^\nu.$$

Consequently,

$$\begin{aligned} R_{\mu\alpha\nu\beta} u^\alpha u^\beta \sum_{i=1}^{2} \xi_i^\mu \xi_i^\nu &= R_{\mu\alpha\nu\beta} u^\alpha u^\beta \left(g^{\mu\nu} + u^\mu n^\nu + n^\mu u^\nu \right) \\ &= R_{\alpha\beta} u^\alpha u^\beta. \end{aligned} \tag{B.8}$$

Indeed, the term containing $u^\mu n^\nu$ vanishes by antisymmetry in μ and α, and the term containing $n^\mu u^\nu$ vanishes by antisymmetry in ν and β.

Thus, in both the timelike and null cases,

$$R_{\mu\alpha\nu\beta} u^\alpha u^\beta \sum_{i} \xi_i^\mu \xi_i^\nu = R_{\alpha\beta} u^\alpha u^\beta.$$

Substitution into (B.6) therefore yields

$$\mathcal{A}_{\mathrm{r}} = -\frac{1}{k} R_{\mu\nu} u^\mu u^\nu, \tag{B.9}$$

where $k = 3$ for a timelike geodesic and $k = 2$ for a null geodesic.

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